Serious Fun with Flexagons
A Compendium and Guide
Serious Fun with Flexagons
Aims and Scope of the Series

The fundamental questions arising in mechanics are: Why?, How?, and How much? The aim of this series is to provide lucid accounts written by authoritative researchers giving vision and insight in answering these questions on the subject of mechanics as it relates to solids.

The scope of the series covers the entire spectrum of solid mechanics. Thus it includes the foundation of mechanics; variational formulations; computational mechanics; statics, kinematics and dynamics of rigid and elastic bodies: vibrations of solids and structures; dynamical systems and chaos; the theories of elasticity, plasticity and viscoelasticity; composite materials; rods, beams, shells and membranes; structural control and stability; soils, rocks and geomechanics; fracture; tribology; experimental mechanics; biomechanics and machine design.

The median level of presentation is the first year graduate student. Some texts are monographs defining the current state of the field; others are accessible to final year undergraduates; but essentially the emphasis is on readability and clarity.
About the Author

Leslie Philip (Les) Pook was born in Middlesex, England in 1935. He obtained a B.Sc. in metallurgy from the University of London in 1956. He started his career at Hawker Siddeley Aviation Ltd., Coventry in 1956. In 1963 he moved to the National Engineering Laboratory, East Kilbride, Glasgow. In 1969, while at the National Engineering Laboratory, he obtained a Ph.D. in mechanical engineering from the University of Strathclyde. Dr. Pook moved to University College London in 1990. He retired formally in 1998 but remained professionally active in the fields of metal fatigue and fracture mechanics, and is affiliated to University College London as a visiting professor. He now has more time to pursue long standing interests in recreational mathematics, including flexagons, and in horology, especially synchronous electric clocks. He is a Fellow of the Institution of Mechanical Engineers and a Fellow of the Institute of Materials, Minerals and Mining. Les married his wife Ann in 1960. They have a daughter, Stephanie, and a son, Adrian.
Preface

Flexagons are rings of hinged polygons that have the intriguing property of displaying different pairs of faces when they are flexed. Workable paper models of flexagons are easy to make and entertaining to manipulate. Flexagons have a surprisingly complex mathematical structure, and just how a flexagon works is not obvious on casual examination of a paper model. The aesthetic appeal of flexagons is in their dynamic behaviour rather than the static appeal of, say, polyhedra. One of the attractions of flexagons is that it is possible to explore their dynamic properties experimentally as well as theoretically. Flexagons may be appreciated at three different levels. Firstly as toys or puzzles, secondly as a recreational mathematics topic, and finally as a subject of serious mathematical study.

My book *Flexagons Inside Out* was published in 2003 by Cambridge University Press. Since then there has been an upsurge in interest in flexagons. Enthusiasts can keep in touch through the *Flexagon Lovers Group*, hosted by Yahoo, and moderated with a light touch by Ann Schwartz. Details of some interesting flexagons have been posted by Group members, and I have enjoyed some stimulating exchanges with other members of the Group. The amount of new information available means that *Flexagons Inside Out* is now outdated. Further geometric analysis has also led to a much better understanding of the behaviour of flexagons, and has in turn led to the discovery of previously unknown flexagons, some of them with entertaining dynamic properties.

Most of the material in the book is new. It is arranged in a logical order appropriate for a textbook on the geometry of flexagons. Extensive cross references are included so that individual chapters do not have to be read in order. Definitions are included in the index so that they can be easily located. It is assumed that the reader already has an interest in flexagons, and has some knowledge of elementary geometry. The book is written so that it can be enjoyed at both the recreational mathematics level, and at the serious mathematics level. In general, detailed proofs are long and tedious, so they are not included. Where there is uncertainty over the accuracy of a conclusion this is made clear in the text. Basic material from *Flexagons Inside Out* is referenced only where needed for clarity but, where appropriate, new material is fully referenced. There are a few errors in *Flexagons Inside Out*, and these are corrected in the present book. In some ways the book is an updated version of the 1962 book length report *Flexagons* by Conrad and Hartline, which is available on the Internet.
A feature of the book is a compendium of over 100 nets for the construction of paper models of some of the more interesting flexagons. These are reproduced at approximately half full size. Many of the nets have not previously been published. The flexagons have been chosen to complement the text, with particular emphasis on demonstrating relationships between different types of flexagon. Three spectacular examples are included. These are the octopus flexagon, the hexa-dodeca-flexagon and the thrice threefold flexagon. Detailed instructions for assembling and manipulating individual flexagons are included for the benefit of those who wish to enjoy flexagons without going into the mathematics. Photographs of some flexagons are included to assist assembly and manipulation.

Most flexigators who move on from making up flexagons from published nets try folding up promising looking nets to see what happens. This bottom up approach has led to the discovery of some interesting flexagons. The top down approach used in this book makes it possible to analyse and understand the dynamic properties of any flexagon. It is also makes it possible to design flexagons having desired properties. Manipulating paper models of the resulting flexagons often reveals unexpected properties that were not predicted theoretically.

January 2009

Les Pook

I do not know how far it is possible to convey to any one who has not experienced it, the peculiar interest, the peculiar satisfaction that lies in a sustained research when one is not hampered by want of money.

H G Wells, *Tono-Bungay*
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1.1 General Features

Flexagons are a twentieth century discovery (Gardner 1965, 2008; Pook 2003). Arthur H Stone, a postgraduate student at Princeton University in America, discovered them in 1939 while folding strips of paper. Figure 1.1a is a photograph of a trihexaflexagon, which was the first type of flexagon to be discovered. The black and white photographs of flexagons in this book are nearly all of models made either from coloured card, coloured origami paper, or from origami duo paper, which is differently coloured on its two surfaces. The appearance of some flexagons is shown as a line diagram such as the ring of four squares shown in Fig. 1.1b.

Workable paper (or card) models of flexagons are easy to make and entertaining to manipulate. They have the intriguing property of displaying different pairs of faces, sometimes in cyclic order, when they are flexed. Flexagons have a surprisingly complex mathematical structure, and just how a flexagon works is not obvious on casual examination of a paper model. The aesthetic appeal of flexagons is in their dynamic behaviour rather than the static appeal of, say, polyhedra. One of the attractions of flexagons is that it is possible to explore their dynamic properties experimentally as well as theoretically. Manipulation of paper models often reveals configurations that have not been predicted theoretically.

A flexagon is a motion structure that has an infinity of states (positions). An umbrella is an everyday example of a motion structure. An edge flexagon consists of a band of identical polygons hinged at common edges by edge hinges. The individual polygons in a flexagon, called leaves, are usually identical (congruent) and are usually regular convex polygons. However, some flexagons consist of other types of convex polygons, and leaves are not always identical. If one hinge of a band is disconnected the band can be laid flat and used as a net to construct a flexagon. Nets are sometimes called templates or friezes. A band of 8 edge hinged squares that has been cut and laid flat as the net for a square even edge flexagon is shown in Fig. 1.2. Assembly and flexing instructions for this flexagon are given in Section 1.4.2. This particular flexagon is a twisted band. This can be seen by disconnecting a hinge...